Optimisation of maintenance policy under parameter uncertainty using portfolio theory

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Abstract

In reliability mathematics, optimisation of maintenance policy is derived based on reliability indexes such as the reliability or its derivatives (e.g., the cumulative failure intensity or the renewal function) and the associated cost information. The reliability indexes, also referred to as *models* in this paper, are normally estimated based on either failure data collected from the field or lab data. The uncertainty associated with them is sensitive to factors such as the sparsity of data. For a company that maintains a number of different systems, developing maintenance policies for each individual system separately and then allocating maintenance budget may not lead to optimal management of the model uncertainty and may lead to cost-ineffective decisions. To overcome this limitation, this paper uses the concept of risk aggregation. It integrates the uncertainty of model parameters in optimisation of maintenance policies and then collectively optimises maintenance policies for a set of different systems, using methods from portfolio theory. Numerical examples are given to illustrate the application of the proposed methods.

Key words: Maintenance, parameter uncertainty, portfolio theory, maintenance policy

1 Introduction

Optimisation of maintenance policies, in particular planning preventive maintenance activities, is an important research topic in reliability mathematics with great relevance to real world applications. It helps the asset managers to monitor the condition of assets and to understand the financial exposure associated with maintenance costs. Many existing methods for maintenance policy optimisation are based on the assumption that the true and precise reliability indexes, also referred to as *models* in this paper, of asset reliability can be obtained. This assumption is unrealistic, in the sense that model uncertainty exists, in particular if model parameters are estimated based on available data. Existing methods for maintenance planning tend to neglect such model uncertainty, which may be particularly problematic if many similar units need to be maintained.

This paper considers maintenance planning under uncertainty about parameter values in an assumed model. The approach proposed is based on the use of sample data to estimate the optimal maintenance policy, which minimizes the expected cost and its variance. The method involves defining a model and estimating its parameters from data. Consequently, estimating the variance for the expected cost (which is a function of the parameters) involves the variance estimated for these parameters.

1.1 Prior work

Maintenance policies are normally developed through applying the least-cost methodology, in which the optimum maintenance policy with the minimised expected cost rate on maintaining a stand-alone system within a given time period is sought (Wang, 2002). To formulate the expected cost rate requires reliability indexes such as the cumulative failure function and the renewal indexes, and cost information such as cost of failure and cost of maintenance (see

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Moghaddass et al. (2015); Zhang & Yang (2015), for example).

Reliability indexes are normally estimated based on lab data and failure data collected from the field (Wu & Scarf, 2015). The sparsity of such data varies from case to case, which can result in different efficiency levels of the maintenance models, or model uncertainty. It is known that three main sources of uncertainties in models stem from (1) data uncertainty (e.g. measurement errors, small sample size, etc); (2) model parameters (uncertainty of model parameters); and (3) model structural uncertainty (models are too simple or too complicated). As identified by Aven (2001), the following two levels of uncertainty exist regarding maintenance models.

Level 1. At the first level, there is uncertainty about observable quantities such as the time to the next failure, for which we develop probability models to describe the uncertainty.

Level 2. At the second level there is uncertainty regarding unobservable quantities such as model structure and model parameters, which stems from the quality of the first level such as the size of observed failure data.

As highlighted by Percy et al. (2010), failure data can be sparse and maintenance scheduling problems are notoriously sensitive to the inaccuracy inherent in model structure and parameter estimates.

In reality, there may be the following scenarios with regard to data collection.

No data scenario. For newly developed or highly reliable equipment, there may be no failure data that can be collected. In this scenario, the technique of expert elicitation can be pursued to estimate the reliability of such equipment. Apparently, uncertainty exists in the probability distributions estimated by the experts. It should be noted that "*elicitation of probability distributions is a far from perfect process*" (O'Hagan & Oakley, 2004) and "*overconfidence is one of the most common (and potentially severe) problems in expert judgement* "(Lin & Bier, 2008).

Sparse data scenario. When the number of observations collected is small, huge uncertainty may exist in the estimated parameters of an assumed probability distribution. In the literature, two main approaches have been developed to tackle this problem. They are the re-sampling technique and the Bayesian updating approach. For example, Laggoune et al. (2010) applies the bootstrap technique to tackle the uncertainties in parameter estimates, considering the problem of the small size of failure data samples. Their work leads to a more general result, as it considers the parameter distribution, according to the available number of failure data. Fang & Huang (2008) uses the Bayesian updating approach to tackling the scenario where data are sparse. Their work, nevertheless, investigates approaches to dealing with model uncertainty for a specific maintenance policy of a given product (system) and may therefore not be ideal for a maintenance planner looking after a set of different types of assets, as argued in the following.

Of course, there is also the scenario of large amounts of data, leading to the possible use of 'big data' methods. This is not addressed in this paper because the typical issues of model uncertainties are very different in this case. Recently, de Jonge et al. (2015) studied the effect of parameter uncertainty on the optimum age-based maintenance strategy. The effect of uncertainty is evaluated by considering both a theoretical uniform lifetime distribution and a more realistic Weibull lifetime distribution.

In the reliability literature, despite presence of uncertainty in maintenance models, maintenance policies are developed to a large degree of mathematical elegance, or "ad hoc", meaning that new maintenance models [2] have been designed in an attempt to represent particular preferences and that little consideration has been paid to the presence of model uncertainty caused by sparsity of data. As a result, most maintenance models and policies, while possessing certain attractive mathematical properties, have been lacking applicability and relevance in the real world.

1.2 Our approach

Since maintenance models possess different levels of uncertainty, it may be cost-ineffective if a company, which maintains a set of different systems, ignores the uncertainty, develops maintenance policy for each individual system separately, and then allocates maintenance budget to each system respectively. A more cost-effective approach may be to collectively optimise maintenance policies for the set of systems, with consideration of model uncertainty. This motivates the research presented in this paper.

[2] In this paper, *maintenance policy* means the frequency of maintenance activities that are executed, and *maintenance model* means the model from which the maintenance policy was developed.

This paper proposes to use portfolio theory, which is widely used in the finance sector (Krokhmal et al., 2011) for managing risk and uncertainty, to collectively optimise maintenance policies for a set of systems. It will concentrate on parameter uncertainty of maintenance models. This uncertainty will be summarised by the variance of the estimated parameters.

The proposed approach can be interesting to both practitioners and researchers. It is useful to the firms that maintain different types of systems but the sparsity of the historical reliability data (e.g., time between failures, or maintenance performance) of different systems may vary, and it can also be applied to optimise a collection of different policies of preventive and predictive maintenance activities. Here, firms can be maintenance companies or manufacturers. For example, a maintenance firm may need to allocate its limited budget on maintenance, or a manufacturer may want to decide the length of the extended warranty for a set of different products sold. In the meantime, academic researchers may be interested to extend this work, for example, some further research topics are listed in the conclusion section of this paper.

To our best knowledge, this is the first paper attempting to collectively optimise different types of maintenance policies (for example, block replacement, age replacement policies, and failure limit policies can be optimised simultaneously) for different systems. The optimization of the maintenance policy through the minimization of the expected cost and its variance is the big novel of the present paper.

1.3 Overview

It should be noted that, in this paper, the term *risk* refers to the financial risk caused by model uncertainty. In reliability mathematics, there are many publications discussing risk-based maintenance (Zhu & Frangopol, 2013; Hu & Zhang, 2014), in which risk, such as risk of health damage or property loss, is caused by system failures. Further, in this paper the term *uncertainty* refers to the "aleatory uncertainty" (or uncertainty due to variability) which represents randomness of samples.

The remainder of this paper is structured as follows. Section 2 formulates the problems considered in this paper. Section 3 discusses properties of the optimisation problems. Section

4 discusses the proposed techniques. Section 5 gives numerical examples on optimisation of maintenance policy. Some concluding remarks are presented in Section 6.

2 Uncertainty of the expected cost

In this section, parameter uncertainty of three maintenance models and their implications are discussed.

2.1 Three replacement policies

In the following, we use three basic replacement policies as examples. Time required for repair or replacement is assumed to be negligible. Of course, the optimisation method proposed in this paper can also be applied to a combination of preventive policies and predictive maintenance policies.

Denote $F(t|\boldsymbol{\theta})$ as the lifetime distribution function, $\bar{F}(t|\boldsymbol{\theta}) = 1 - F(t|\boldsymbol{\theta})$, and $\boldsymbol{\theta}$ as the parameter vector of $F(t|\boldsymbol{\theta})$ (for example, $\boldsymbol{\theta}$ in the Weibull distribution includes both the shape parameter and the location parameter). Let $M(t|\boldsymbol{\theta}) = \sum_{n=1}^{\infty} F^{(n)}(t|\boldsymbol{\theta})$, where $F^{(n)}(t|\boldsymbol{\theta})$ is the nth Stieltjes convolution of $F(t|\boldsymbol{\theta})$ and $F^{(n)}(t|\boldsymbol{\theta}) = \int_0^t F^{(n-1)}(t-u|\boldsymbol{\theta})dF(u|\boldsymbol{\theta})$.

We consider the following three maintenance policies, with the optimality criteria all based on the renewal reward theory (Barlow & Proschan, 1965). One could also include different criteria (Coolen-Schrijner & Coolen, 2006), this is left as a topic for future research.

Age replacement Under this policy, one replaces a unit with a new identical unit either at a pre-determined age T or upon failure if it occurs earlier. Let c_1 be the cost of replacement for a failed unit and c_2 be the cost of scheduled replacement. Then according to Barlow & Proschan (1965), the expected cost is given by

$$C_a(T|\boldsymbol{\theta}) = \frac{c_1 F(T|\boldsymbol{\theta}) + c_2 \bar{F}(T|\boldsymbol{\theta})}{\int_0^T \bar{F}(t|\boldsymbol{\theta})dt}. \tag{1}$$

Block replacement Under this policy, a unit is replaced with a new identical unit upon failure or at fixed times $kT (k = 1, 2, ...,)$. Let c_1 be the cost of replacement for a failed unit and c_2 be the cost of scheduled replacement. Then according to Barlow & Proschan

(1965), the expected cost is given by

$$C_b(T|\boldsymbol{\theta}) = \frac{c_1 M(T|\boldsymbol{\theta}) + c_2}{T}. \tag{2}$$

Periodic replacement with minimal repair Under this policy, a unit is replaced at fixed times $kT(k = 1, 2, ...)$ with a new identical unit. If a unit fails between replacements, minimal repair is made to restore the unit back to work. Denote c_1 as the cost of minimal repair and c_2 as the cost of scheduled replacement. Then according to Barlow & Proschan (1965), the total expected cost is given by

$$C_p(T|\boldsymbol{\theta}) = \frac{c_1 \Lambda(T|\boldsymbol{\theta}) + c_2}{T}, \tag{3}$$

where $\Lambda(x|\boldsymbol{\theta})$ is the cumulative intensity function of the repairable unit.

Note that it is reasonable to assume that $c_1 > c_2$. This assumption will be followed in this paper.

2.2 Uncertainty of the expected cost

As discussed in Section 1.1, the uncertainty of a model is larger (smaller) if the model is developed based on a smaller (larger) sample size. That is, for example, in the equations (1), (2), and (3), the variances of estimated parameters $\boldsymbol{\theta}$ are larger for small sample size than those for large sample size. Now the following two questions are important.

Q1. how to estimate the variance for parameters $\boldsymbol{\theta}$; and
Q2. how to estimate the variance for a function of parameters $\boldsymbol{\theta}$.

Different methods for Q1 can be found in the literature. For example, in case of fitting the Weibull distribution, one may have three scenarios: (1) when there is no data available or a very small sample size ($n < 10$ for example) for developing a model, expert elicitation (EE) or EE with Bayesian updating may be pursued to estimate the variance (Speirs-Bridge et al., 2010); (2) when the sample size n is between 10 and 52, the weighted least-square method is preferable (Wu et al., 2006); and (3) If one has a larger sample size ($n \geq 53$), then the maximum likelihood may be used (Wu et al., 2006).

To answer Q2, one may use one of the following methods to estimate the variances of $C_a(T|\boldsymbol{\theta})$,

$C_b(T|\boldsymbol{\theta})$, and $C_p(T|\boldsymbol{\theta})$ after the variance of $\boldsymbol{\theta}$ is given. That is, one needs to estimate the variance for $g(\boldsymbol{\theta})$ if the variances for $\boldsymbol{\theta}$ is given, here $g(\boldsymbol{\theta}) = C_a(T|\boldsymbol{\theta})$ for the age replacement policy, for example.

In the literature, there are some approaches to estimating confidence intervals for functions of parameters, for example, the bootstrap method Krinsky & Robb (1986) and the δ method (see p. 172, Weisberg (2014)).

2.3 Decision problem

Suppose a company is maintaining n different systems and attempting to develop maintenance policies for those systems. Denote $C_i(T_i|\boldsymbol{\theta}_i)$ as the expected cost of the maintenance policy for system i ($i = 1, 2, ..., n$), where T_i is the vector of decision variables such as preventive maintenance intervals for maintaining system i, and $\boldsymbol{\theta}_i$ includes the parameters in the reliability index of system i.

Then, the expected total cost of maintenance of the n systems is given by

$$\mathrm{C}(\boldsymbol{T}|\boldsymbol{\theta}) = \sum_{i=1}^{n} C_i(T_i|\boldsymbol{\theta}_i), \tag{4}$$

where $\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \dots, \boldsymbol{\theta}_n)$ and $\boldsymbol{T} = (T_1, \dots, T_n)$.

In the literature, the existing approach to optimising maintenance policy is to minimise each $C_i(T_i; \boldsymbol{\theta}_i)$, respectively, and then to allocate the total amount to each of the n systems. However, as $\boldsymbol{\theta}_i$ is estimated based on observations, its efficiency can vary with sample sizes. In other words, $\boldsymbol{\theta}_i$ can be treated as a random variable. Consequently, the total amount, $\mathrm{C}(\boldsymbol{T}|\boldsymbol{\theta})$, becomes a random variable. This leads to different attitudes of the decision makers in order to deal with the possible variation in expected costs resulting from uncertainty about the parameter value. For example, assume that a company is maintaining a number of systems. Suppose that there are several options of maintenance policies are designed and they have different total expected cost $\mathrm{C}(\boldsymbol{T}|\boldsymbol{\theta})$ and different variance $\mathrm{V}_{\boldsymbol{\theta}}(\mathrm{C}(\boldsymbol{T}|\boldsymbol{\theta}))$. Then, the selection of options may depend on the company's risk attitude as well as its budget. (1) If the company is risk-averse, it may choose an option with lower risk (i.e., $\mathrm{V}_{\boldsymbol{\theta}}(\mathrm{C}(\boldsymbol{T}|\boldsymbol{\theta}))$); (2) if the company is risk-prone, it may choose an option having lower expected cost (i.e., $\mathrm{C}(\boldsymbol{T}|\boldsymbol{\theta})$).

That is, to choose an option depends not only on the total expected cost $\mathrm{C}(\boldsymbol{T}|\boldsymbol{\theta})$, but also on the variance $\mathrm{V}_{\boldsymbol{\theta}}(\mathrm{C}(\boldsymbol{T}|\boldsymbol{\theta}))$. Of course, the total attitude towards risk can be taken into account, for example through the use of utility functions. However, this is often difficult to measure meaningfully, in portfolio theory the variance is typically used as a surrogate measure for risk. In other words, company's risk attitude plays an important role in decision making. Risk-averse planners differ from risk-loving planners in that they are concerned more about the decisions that minimise risk than about those that minimise the expected cost.

In summary, factors influencing decision making include the total expected cost, decision maker's risk attitude, and available budget. Those factors will be considered when we optimise maintenance policies in the following sections.

3 Collective optimisation of risk

In this section, we propose three new methods to optimise maintenance policies.

3.1 Methods of collectively optimising PM policies

To incorporate risk in optimisation of maintenance policies, one may choose one of the following three optimisation problems.

Option 1. One may explicitly trade risk against the expected cost in the objective function:

$$\min_{\boldsymbol{T}\in\mathbb{R}} \mathrm{C}(\boldsymbol{T}|\boldsymbol{\theta}) + \rho\, \mathrm{V}_{\boldsymbol{\theta}}(\mathrm{C}(\boldsymbol{T}|\boldsymbol{\theta})). \tag{5}$$

where ρ is a pre-specified non-negative parameter and it is typically assumed to be restricted to values in $[0, 1]$. The value of ρ reflects the decision maker's risk attitude. For example, if the decision maker is risk averse, ρ may be set larger. In the current formulation ρ would not be unit-less, its unit would be the chosen monetary unit to the power -1. Note that alternative formulations replace the variance by the standard deviation, or divide both the expected costs and the variance by similar functions, for example corresponding to a standard scenario, in order to do away with units and hence to make the sum meaningful. In all cases though, this option relates to an explicit weighting of expected costs and risk which are together minimised. Clearly, this can also be considered as only allowing solutions on a Pareto boundary, that is any decrease in expected costs at an optimal value will lead to increase of the risk.

Option 2. One may select the decision $\boldsymbol{T}$ that minimises the expected cost $\mathrm{C}(\boldsymbol{T}|\boldsymbol{\theta})$ while imposing a maximum level of risk $\mathrm{V}_{\boldsymbol{\theta}}(.)$:

$$\min_{\boldsymbol{T}\in\mathbb{R}} \mathrm{C}(\boldsymbol{T}|\boldsymbol{\theta}), \qquad (6)$$
$$\text{subject to} : \mathrm{V}_{\boldsymbol{\theta}}(\mathrm{C}(\boldsymbol{T}|\boldsymbol{\theta})) \leq \nu_0.$$

where ν_0 reflects decision maker's risk attitude. For example, if the decision maker is risk averse, ν_0 may be set small.

Option 3. A third alternative can be to minimise the risk $\mathrm{V}_{\boldsymbol{\theta}}(.)$ while imposing a maximum level of the expected cost $\mathrm{C}(\boldsymbol{T}|\boldsymbol{\theta})$:

$$\min_{\boldsymbol{T}\in\mathbb{R}} \mathrm{V}_{\boldsymbol{\theta}}(\mathrm{C}(\boldsymbol{T}|\boldsymbol{\theta})), \qquad (7)$$
$$\text{subject to} : \mathrm{C}(\boldsymbol{T}|\boldsymbol{\theta}) \leq C_0.$$

where C_0 reflects the budget constraint.

Option 3 is mainly used for portfolio management in finance. However, it is thought that its inclusion is relevant as practical maintenance planning under quite severe budget limitations is very common, with maintenance managers often explicitly working towards using an allocated budget while aiming to keep variance low, a reflection of low risk of huge overspend.

In the following, we use letters a, b and p in the subscripts of relevant quantities to distinguish notation of the age replacement policy, the block replacement policy and the periodic replacement with minimal repair policy, respectively.

Suppose a set of one-unit systems composed of m_a systems maintained with age replacement policy, m_b systems maintained with block replacement policy, and m_p systems maintained with periodic replacement with minimal repair policy. We assume throughout that all m_a systems subject to age replacement are exchangeable with regard to their time to failure and costs associated with running and maintaining them. Of course, in practice this may not be entirely realistic. The method presented in this paper can be quite easily adapted to more complicated scenarios, studying such scenarios from practical perspective is an important topic for future research.

We now have the following three properties.

In the following, we assume that the parameters in maintenance policies for different systems are statistically independent.

Property 1 *With the optimisation Option 1, the objective function is given by*

$$G(\boldsymbol{T}) = m_a C_a(T_a|\boldsymbol{\theta}_a) + m_b C_b(T_b|\boldsymbol{\theta}_b) + m_p C_p(T_p|\boldsymbol{\theta}_p) + \rho\left(m_a^2 \mathrm{V}_{\boldsymbol{\theta}}(C_a(T_a|\boldsymbol{\theta}_a)) + m_b^2 \mathrm{V}_{\boldsymbol{\theta}}(C_b(T_b|\boldsymbol{\theta}_b)) + m_p^2 \mathrm{V}_{\boldsymbol{\theta}}(C_p(T_p|\boldsymbol{\theta}_p))\right). \tag{8}$$

Assume $f_a(T_a|\boldsymbol{\theta}_a)/\bar{F}_a(T_a|\boldsymbol{\theta}_a)$ is continuous and increasing with respect to T_a. Then there exists an optimum solution $\boldsymbol{T}^* = (T_a^*, T_b^*, T_p^*)$ that minimises the following problem:

$$\min_{\boldsymbol{T}\in\mathbb{R}} G(\boldsymbol{T}). \tag{9}$$

The proof of the properties can be found in the Appendix. We should emphasize here that the variance of the expected costs is due to the assumed uncertainty in the parameter value. At planning stage, the expected costs for all systems that are maintained with age replacement are the same, and hence all these systems will be maintained with the use of the same optimal value for T_a. Hence, when we take the variance of the sum of these m_a expected costs, we get the term m_a^2 times the variance expression as given above.

Property 2 *With the optimisation Option 2, the objective function is given by*

$$\min_{\boldsymbol{T}\in\mathbb{R}} m_a C_a(T_a|\boldsymbol{\theta}_a) + m_b C_b(T_b|\boldsymbol{\theta}_b) + m_p C_p(T_p|\boldsymbol{\theta}_p), \tag{10}$$

$$\text{subject to}: m_a^2 \mathrm{V}_{\boldsymbol{\theta}}(C_a(T_a|\boldsymbol{\theta}_a)) + m_b^2 \mathrm{V}_{\boldsymbol{\theta}}(C_b(T_b|\boldsymbol{\theta}_b)) + m_p^2 \mathrm{V}_{\boldsymbol{\theta}}(C_p(T_p|\boldsymbol{\theta}_p)) \leq \nu_0. \tag{11}$$

Then, one can use the following algorithm to find the optimum PM policies.

Algorithm 2. *The optimisation process under Option 2 can be done with the following procedures.*

Step 1. Find the maintenance policy for each system respectively, and then check if the constraint in (11) holds. If the constraint holds, then stop; otherwise, go to Step 2;

Step 2. Solve equations (12–15) to obtain optimum solutions.

$$\frac{\partial C_a(T_a|\boldsymbol{\theta}_a)}{\partial T_a} + \mu m_a \frac{\partial \mathrm{V}_{\boldsymbol{\theta}}(C_a(T_a|\boldsymbol{\theta}_a))}{\partial T_a} = 0, \tag{12}$$

$$\frac{\partial C_b(T_b|\boldsymbol{\theta}_b)}{\partial T_b} + \mu m_b \frac{\partial \mathrm{V}_{\boldsymbol{\theta}}(C_a(T_b|\boldsymbol{\theta}_b))}{\partial T_b} = 0, \tag{13}$$

$$\frac{\partial C_p(T_p|\boldsymbol{\theta}_p)}{\partial T_p} + \mu m_p \frac{\partial \mathrm{V}_{\boldsymbol{\theta}}(C_p(T_p|\boldsymbol{\theta}_p))}{\partial T_p} = 0, \tag{14}$$

$$\mu(m_a^2 \mathrm{V}_{\boldsymbol{\theta}}(C_a(T_a|\boldsymbol{\theta}_a)) + m_b^2 \mathrm{V}_{\boldsymbol{\theta}}(C_b(T_b|\boldsymbol{\theta}_b)) + m_p^2 \mathrm{V}_{\boldsymbol{\theta}}(C_p(T_p|\boldsymbol{\theta}_p)) - \nu_0) = 0, \tag{15}$$

where μ is a Lagrange multiplier.

Property 3 *With the optimisation Option 3, the objective function is given by*

$$\min_{\boldsymbol{T}\in\mathbb{R}} m_a^2 \mathrm{V}_{\boldsymbol{\theta}}(C_a(T_a|\boldsymbol{\theta}_a)) + m_b^2 \mathrm{V}_{\boldsymbol{\theta}}(C_b(T_b|\boldsymbol{\theta}_b)) + m_p^2 \mathrm{V}_{\boldsymbol{\theta}}(C_p(T_p|\boldsymbol{\theta}_p)), \tag{16}$$

$$\text{subject to}: m_a C_a(T_a|\boldsymbol{\theta}_a) + m_b C_b(T_b|\boldsymbol{\theta}_b) + m_p C_p(T_p|\boldsymbol{\theta}_p) \leq C_0. \tag{17}$$

Then, one can use the following algorithm to find the optimum PM policies.

Algorithm 3. *The optimisation process under Option 3 can be done with the following procedures.*

Step 1. If C_0 is large enough, the optimum maintenance policy is that no replacement should not be conducted; otherwise, go to step 2;

Step 2. One can solve equations (18–21) to obtain optimum solutions.

$$\mu \frac{\partial C_a(T_a|\boldsymbol{\theta}_a)}{\partial T_a} + m_a \frac{\partial \mathrm{V}_{\boldsymbol{\theta}}(C_a(T_a|\boldsymbol{\theta}_a))}{\partial T_a} = 0 \tag{18}$$

$$\mu \frac{\partial C_b(T_b|\boldsymbol{\theta}_b)}{\partial T_b} + m_b \frac{\partial \mathrm{V}_{\boldsymbol{\theta}}(C_a(T_b|\boldsymbol{\theta}_b))}{\partial T_b} = 0 \tag{19}$$

$$\mu \frac{\partial C_p(T_p|\boldsymbol{\theta}_p)}{\partial T_p} + m_p \frac{\partial \mathrm{V}_{\boldsymbol{\theta}}(C_p(T_p|\boldsymbol{\theta}_p))}{\partial T_p} = 0 \tag{20}$$

$$\mu(m_a C_a(T_a|\boldsymbol{\theta}_a) + m_b C_b(T_b|\boldsymbol{\theta}_b) + m_p C_p(T_p|\boldsymbol{\theta}_p) - C_0) = 0 \tag{21}$$

where μ is a Lagrange multiplier.

The equations in Step 2 in both Algorithm 2 and Algorithm 3 follow from the well-known Karush-Kuhn-Tucker first-order necessary condition for a candidate solution point to be an optimum solution. Those conditions are not sufficient. To seek the sufficient conditions, or the second order sufficient conditions, the reader is referred to Theorem 4.46 in page 99 in Zörnig (2014), which will not be repeated in the current paper.

To apply the above-proposed methods in optimisation problems shown in Eqs (5)–(7), one needs to calculate $\mathrm{V}_{\boldsymbol{\theta}}(C_a(T_a|\boldsymbol{\theta}_a))$, $\mathrm{V}_{\boldsymbol{\theta}}(C_b(T_b|\boldsymbol{\theta}_b))$ and $\mathrm{V}_{\boldsymbol{\theta}}(C_p(T_p|\boldsymbol{\theta}_p))$. Below we give some example methods of calculating those quantities for the case when the lifetime distribution is the Weibull distribution.

Calculation of $\mathrm{V}_{\boldsymbol{\theta}}(C_a(T_a|\boldsymbol{\theta}_a))$.

Given lifetime distribution $F(t|\boldsymbol{\theta}) = 1 - \exp\left(-\frac{t^{\beta_a^{-1}}}{\alpha_a}\right)$. Assume m_a units are under a Type I censoring scheme: unit i may fail at X_i before the threshold time τ_a or survive over the time interval $(0, \tau_a)$. From Sirvanci & Yang (1984), the estimator of α_a and its variance are given by

$$\hat{\alpha_a} = \sum_{i=1}^{m_a} \frac{V_i^{\beta_a^{-1}}}{M}, \tag{22}$$

and

$$\mathrm{V}(\hat{\alpha_a}) = \frac{\alpha_a^2}{p} + \frac{\alpha_a^2}{\beta_a^2}(1 + I_{1p} + \log\alpha_a)^2 \mathrm{V}(\beta_a), \tag{23}$$

respectively, where $V_i = \min\{X_i, \tau_a\}$, M is the number of failures occurring in $[0, \tau_a]$, p can be estimated by $\hat{p} = \frac{M}{m_a}$, $I_{1p} = \frac{1}{p}\int_0^p \log\log\frac{1}{1-t}dt$, and $h(p) = \log\log\frac{1}{1-p} - I_{1p}$.

The estimator of β_a and its variance are given by

$$\hat{\beta}_a = \sum_{i=1}^{m_a} \frac{(\log\tau_a - \log X_i)I_i}{Mh(p)}, \tag{24}$$

and

$$\mathrm{V}(\hat{\beta}_a) = \frac{\beta_a^2 I_{2p} - \beta_a^2 p I_{1p}^2 + \beta_a^2 p^3(1-p)\left\{\left[(1-p)\log\frac{1}{1-p}\right]^{-1} - \frac{h(p)}{p}\right\}^2}{(ph(p))^2}, \tag{25}$$

respectively, where I_i is the indicator of the event $[X_i < \tau_a]$. That is, $I_i = 1$ if item i fails in $[0, \tau_a]$ and $I_i = 0$ otherwise, and $I_{2p} = \int_0^p \left(\log\log\frac{1}{1-t}\right)^2 dt$.

The covariance of $\hat{\beta}_a$ and $\hat{\alpha_a}$ is given by

$$\mathrm{cov}(\hat{\alpha_a}, \hat{\beta}_a) = \frac{1}{\beta_a^2}\left[-\alpha_a\beta_a(1 + I_{1p} + \log\alpha_a)\mathrm{V}(\beta_a)\right]. \tag{26}$$

One may use the δ method to calculate the variance of the expected cost for the age replacement policy.

Calculation of $\mathrm{V}_{\boldsymbol{\theta}}(C_b(T_b|\boldsymbol{\theta}_b))$.

Given lifetime distribution $F(t|\boldsymbol{\theta}) = 1 - \exp\left[-\left(\frac{t^{\beta_b^{-1}}}{\alpha_b}\right)\right]$, in order to calculate the variance of $\mathrm{V}_{\boldsymbol{\theta}}(C_b(T_b|\boldsymbol{\theta}_b))$, one needs first to obtain the renewal function $M(t)$. Assume m_b units are under a Type I censoring scheme: unit i may fail at X_i before the threshold time τ_b or survive over the time interval $(0, \tau_b)$. Jiang (2010) develops the following formula to approximate $M(t|\boldsymbol{\theta})$,

$$M(t|\boldsymbol{\theta}) = q\left[1 - \exp\left[-\left(\frac{t^{\beta_b^{-1}}}{\alpha_b}\right)\right]\right] + (1-q)\left(\frac{t^{\beta_b^{-1}}}{\alpha_b}\right), \tag{27}$$

where $q = 1 - \exp\left[-\frac{1}{0.8818}\left(\frac{1-\beta_b}{\beta_b}\right)^{0.9269}\right]$. Based on the above approximation of $M(t)$ and similar estimators as those shown in Eqs. (22)—(25)[3], one can use the δ method to calculate the variance of $\mathrm{V}_{\boldsymbol{\theta}}(C_b(T_b|\boldsymbol{\theta}_b))$ for the block replacement policy.

Calculation of $\mathrm{V}_{\boldsymbol{\theta}}(C_p(T_p|\boldsymbol{\theta}_p))$.

Calculating $\mathrm{V}_{\boldsymbol{\theta}_\mathrm{p}}(C_p(T_p|\boldsymbol{\theta}_p))$ differs from calculating $\mathrm{V}_{\boldsymbol{\theta}_\mathrm{a}}(C_p(T_a|\boldsymbol{\theta}_a))$ and $\mathrm{V}_{\boldsymbol{\theta}_\mathrm{b}}(C_p(T_b|\boldsymbol{\theta}_b))$, as there exists minimal repair between replacements and therefore involves the estimation of the intensity function $\Lambda(t)$. Assume there are m_p identical repairable systems. These systems are tested until time τ_p and the failure times of these systems are recorded as $\{t_{1,1}, \ldots, t_{1,n_1}\}, \ldots, \{t_{m_p,1}, \ldots, t_{m_p,n_{m_p}}\}$. We assume that the failure process in Section 2 follows a power law NHPP (Non-Homogeneous Poisson Process) model. That is, the expected number of failures and failure intensity function for the system are derived from the following equation:

$$\lambda(t) = \alpha_p \beta_p t^{\beta_p - 1}. \tag{28}$$

Then, the expected number of failures at time t is:

$$\Lambda(t) = \alpha_p t^{\beta_p^{-1}}. \tag{29}$$

$\Lambda(t)$ can be estimated from failure data based on the following approach, as indicated in Guo & Pan (2008). The variance of $\mathrm{V}[\hat{\Lambda}(t)]$ is given by Guo & Pan (2008)

$$\mathrm{V}[\hat{\Lambda}(t)] = \frac{\alpha_p t^{2\beta_p^{-1}}}{m_p \tau_p^{\beta_p^{-1}}}\left[1 + \beta_p^{-2}(\ln\tau_p - \ln t)^2\right], \tag{30}$$

where $\hat{\alpha_p} = \dfrac{\sum_{j=1}^{m_p} n_j}{m_p \tau_p^{\hat{\beta_p}^{-1}}}$ and $\hat{\beta_p} = \dfrac{\sum_{j=1}^{m_p} n_j \ln \tau_p - \sum_{j=1}^{m_p}\sum_{i=1}^{n} \ln t_{ij}}{\sum_{j=1}^{m_p} n_j}$.

Thus, the variance of $C_p(T_p|\boldsymbol{\theta}_p)$ is given by

$$\mathrm{V}_{\boldsymbol{\theta}}(C_p(T_p|\boldsymbol{\theta}_p)) = \left\{\frac{c_1^2 \alpha_p T_p^{2\beta_p^{-1}-2}}{m_p \tau_p^{\beta_p^{-1}}}\left[1 + \beta_p^{-2}(\ln\tau_p - \ln T_p)^2\right]\right\}^{1/2}. \tag{31}$$

[3] One can simply replace m_a with m_b in (22)—(25) to obtain the estimators of α_b and β_b.

4 Discussion

The above-mentioned three reliability indexes, the reliability function, the renewal function, and the cumulative hazard function, are the most frequently used basic reliability indexes in developing maintenance policies. The above methods simply offer examples how the variances of the expected cost rate can be calculated, when the uncertainty about the parameter estimates is taken into account. In real-world applications and from a research perspective, other maintenance policies can also be optimised jointly.

In each of the three maintenance policies discussed above, we assume systems are identical. Normally, a maintenance policy is assumed to be optimised on a set of identical items. In case there are a lot of different items, then different maintenance policies should be applied on them but the method proposed in this paper, which is to jointly optimise maintenance policies for all the items, are still applicable.

Developing maintenance policy with the above-proposed approach requires estimating the variances of given reliability indexes $g(\boldsymbol{\theta})$, which can be the renewal function or the cumulative hazard function. It should be noted that, however, for a given reliability index, using different estimation methods to derive the variances can result in different levels of efficiency. In other word, the selection of estimation methods provides another source of uncertainty caused. For example, using a parametric method may yield smaller variance than using a nonparametric method (From & Tortorella, 2005).

5 Numerical examples

In this section, we illustrate optimisation of preventive maintenance policy along the lines presented in this paper, using the bootstrap method and the δ method to deal with the uncertainty due to the parameter estimation.

The choice of the bootstrap method or the δ method depends on sample size. The bootstrap method may be used if the sample size is small (see Krinsky & Robb (1986)), and the δ method may be used if the sample size is large (see p. 172, Weisberg (2014)). Below a short introduction to the two methods is given.

5.1 A short introduction to the bootstrap method and the δ method

To answer Q2 listed in Section 2.2, one may use one of the following methods to estimate the variances of $C_a(T_a|\boldsymbol{\theta}_a)$, $C_b(T_b|\boldsymbol{\theta}_b)$, and $C_p(T_p|\boldsymbol{\theta}_p)$ after the variance of $\boldsymbol{\theta}$ is given. That is, one needs to estimate the variance for $g(\boldsymbol{\theta})$ if the variances for $\boldsymbol{\theta}$ is given, where $g(\boldsymbol{\theta})$ may be $C_a(T_a|\boldsymbol{\theta}_a)$ for the age replacement policy, for example.

The bootstrap method. For example, Krinsky and Robb Krinsky & Robb (1986) recommend the following procedure to obtain a $(1-\alpha)$ confidence interval for $g(\boldsymbol{\theta})$. On the basis of this method, one can also estimate the variance:

Step 1. generate a large number of random numbers from the distribution of the parameter estimates;

Step 2. calculate the function value $g(.)$ for each random number; and

Step 3. trim $(\alpha/2)$ from each tail of the resulting distribution of the function values.

The δ method. When the sample size is big, if the function $g(.)$ is differentiable and has nonzero and bounded derivatives, one may use the following delta method (see p. 172, Weisberg (2014)).

In probability theory, if a consistent estimator $\boldsymbol{X}$ converges in probability to its true value $\boldsymbol{\theta}$, the asymptotic normality can be obtained using the central limit theorem:

$$\sqrt{n_0}\,(X-\boldsymbol{\theta}) \xrightarrow{D} N\,(0,\Sigma)\,, \tag{32}$$

where n_0 is the number of observations, $\xrightarrow{D}$ denotes convergence in distribution, and $\boldsymbol{\Sigma}$ is a (symmetric positive semi-definite) covariance matrix. Then the delta method implies that

$$\sqrt{n_0}\,(g(\boldsymbol{X})-g(\boldsymbol{\theta})) \xrightarrow{D} N\left(0,\nabla g(\boldsymbol{\theta})^T\cdot\Sigma\cdot\nabla g(\boldsymbol{\theta})\right). \tag{33}$$

5.2 Parameter settings

The following parameter settings are used in this subsection. The parameters in Table 1 are arbitrarily selected. In a real situation, some of these would be estimated from data, and some would be predetermined by the manufacturing team.

In Table 1, we use c_{1*} and c_{2*} to distinguish repair costs that occurred in age replacement,

Table 1: Parameter setting.

α_a	β_a	τ_a	c_{1a}	c_{2a}	α_b	β_b	τ_b	c_{1b}	c_{2b}	α_p	β_p	τ_p	c_{1p}	c_{2p}
1	0.5	1.25	20	10	1	0.333	1	25	10	1	0.5	2	40	10

block replacement, and periodic replacement policies. For example, c_{1a} and c_{2a} are the cost of replacement for a failed item and the cost of replacement for a scheduled item, respectively. Here, we assume that $c_{1*} > c_{2*}$ as replacement due to an unexpected failure typically incurs higher costs than scheduled replacement.

5.3 *Using the bootstrap method*

We assume $m_a = m_b = m_p = 10$.

We use the bootstrap method proposed by Efron (1981). The idea of this method is to sample data from the censored dataset and the uncensored dataset respectively, and then combine the two sample sets as a new sample set. We obtain the estimated parameters with the given data and plot the cost and variance variation with respect to the replacement time T_a, T_b, and T_p.

5.3.1 *Independently optimising maintenance policies*

Age replacement The results are $\hat{\alpha}_a = 0.942$ and $\hat{\beta}_a = 0.335$. When $T_a = 0.80$, the optimal cost $C_a(T|\boldsymbol{\theta}) = 20.1$ is achieved and its associated variance $Var(C_a(T|\boldsymbol{\theta})) = 2.02$.

Block replacement The results are $\hat{\alpha}_b = 1.0395$ and $\hat{\beta}_b = 0.341$. When $T_b = 0.66$, the optimal cost $C_b(T|\boldsymbol{\theta}) = 24.7$ is achieved and its associated variance $Var(C_a(T|\boldsymbol{\theta})) = 8.37$.

Periodic replacement with minimal repair The results are $\hat{\alpha}_p = 0.753$ and $\hat{\beta}_p = 0.567$. When $T_p = 0.63$, the optimal cost $C_p(T|\boldsymbol{\theta}) = 37.4$ is achieved and its associated variance $Var(C_a(T|\boldsymbol{\theta})) = 9.6$.

5.3.2 *Collectively optimising maintenance policies*

For the three options, we have the following results.

Option 1 Let $\rho = 0.2$. When $T_a = 0.81$, $T_b = 1.19$ and $T_p = 0.61$, the optimal solution

$G(\boldsymbol{T}) = 1076.2$ is achieved.

Option 2 On the basis of the results of the three policies in Section 5.3.1, the variance in the left hand-side of the inequality (11): $m_a^2 \mathrm{V}_{\boldsymbol{\theta}}(C_a(T_a|\boldsymbol{\theta}_a)) + m_b^2 \mathrm{V}_{\boldsymbol{\theta}}(C_b(T_b|\boldsymbol{\theta}_b)) + m_p^2 \mathrm{V}_{\boldsymbol{\theta}}(C_p(T_p|\boldsymbol{\theta}_p)) = 100 \times (2.02 + 8.37 + 9.6) = 1999$, which implies the following two situations.

- if ν is set to be smaller than 1999, the constraint in the inequality (11) should take effect on the objective function;

  Let $\nu_0 = 1744$. When $T_a = 0.81$, $T_b = 0.63$ and $T_p = 0.63$, the optimal solution $G(\boldsymbol{T}) = 825$ is achieved. As can be seen, the values T_a, T_b and T_p are not equal to those values in the three policies in Section 5.3.1. This implies that the constraint takes effect.

- if ν is set to be larger than 1999, the constraint in the inequality (11) should not take effect on the objective function.

  If we let $\nu_0 = 2044$. When $T_a = 0.81$, $T_b = 0.68$ and $T_p = 0.63$, the optimal solution $G(\boldsymbol{T}) = 822$ is achieved. As can be seen, the values T_a, T_b and T_p are the same values as in the three policies in Section 5.3.1. This implies that the constraint does not take effect.

Option 3 Let $C_0 = 822$. When $T_a = 0.81$, $T_b = 0.68$ and $T_p = 0.63$, the optimal solution $G(\boldsymbol{T}) = 1944$ is achieved.

Let $C_0 = 922$. When $T_a = 0.81$, $T_b = 1.99$ and $T_p = 0.42$, the optimal solution $G(\boldsymbol{T}) = 1095$ is achieved.

5.4 Using the δ method

We set $m_a = m_b = m_p = 200$, so we are considering maintenance policies for a large organisation which applies the same policies to many systems. Of course, the uncertainties due to parameter estimation in assumed mathematical models will have a more substantial influence the larger the number of systems maintained is.

5.4.1 Independently optimising maintenance policies

Age replacement The results are $\hat{\alpha}_a = 1.053$, $V(\hat{\alpha}_a) = 1.419$, $\hat{\beta}_a = 0.469$, $V(\hat{\beta}_a) = 0.219$, and covariance between parameters α_a and β_a is -0.0383. When $T_a = 1.03$, the optimal cost $C_a(T|\boldsymbol{\theta}) = 21.05$ is achieved and its associated variance $Var(C_a(T|\boldsymbol{\theta})) = 139.5$.

Block replacement The results are $\hat{\alpha}_b = 1.0226$, $V(\hat{\alpha}_v) = 1.774$, $\hat{\beta}_b = 0.271$, $V(\hat{\beta}_b) =$

0.112, and covariance between parameters α_b and β_b is 0.107. When $T_b = 0.63$, the optimal cost $C_b(T|\boldsymbol{\theta}) = 22.37$ is achieved and its associated variance $Var(C_b(T|\boldsymbol{\theta})) = 216.00$.

Periodic replacement with minimal repair The results are $\hat{\alpha}_p = 1.0385$ and $\hat{\beta}_p = 1.981$. When $T_p = 0.49$, the optimal cost $C_p(T|\boldsymbol{\theta}) = 41.04$ is achieved and its associated variance $Var(C_a(T|\boldsymbol{\theta})) = 2.13$.

5.4.2 Collectively optimising maintenance policies

On the three options, we have the following results.

Option 1 Let $\rho = 0.001$. When $T_a = 0.84$, $T_b = 1.38$ and $T_p = 0.49$, the optimal solution $G(\boldsymbol{T}) = 25220$ is achieved.

Option 2 On the basis of the results of the three policies in Section 5.3.1, the variance in the left hand-side of the inequality (11): $m_a^2 \mathrm{V}_{\boldsymbol{\theta}}(C_a(T_a|\boldsymbol{\theta}_a)) + m_b^2 \mathrm{V}_{\boldsymbol{\theta}}(C_b(T_b|\boldsymbol{\theta}_b)) + m_p^2 \mathrm{V}_{\boldsymbol{\theta}}(C_p(T_p|\boldsymbol{\theta}_p)) = 40000 \times (139.5 + 216.0 + 2.13) = 1.43052 \times 10^7$, which implies the following two situations.

- if ν is set to be smaller than 1.43052×10^7, the constraint in the inequality (11) should take effect on the objective function;

  Let $\nu_0 = 1.23 \times 10^7$. When $T_a = 1.01$, $T_b = 0.54$ and $T_p = 0.59$, the optimal solution $G(\boldsymbol{T}) = 17011$ is achieved. As can be seen, the values T_a, T_b and T_p are not equal to those values in the three policies in Section 5.3.1. This implies that the constraint takes effect.

- if ν is set to be larger than 1.43052×10^7, the constraint in the inequality (11) should not take effect on the objective function.

  Let $\nu_0 = 1.53 \times 10^7$. When $T_a = 1.03$, $T_b = 0.63$ and $T_p = 0.49$, the optimal solution $G(\boldsymbol{T}) = 16891$ is achieved. As can be seen, the values T_a, T_b and T_p are the same those values in the three policies in Section 5.3.1. This implies that the constraint does not take effect.

Option 3 Let $C_0 = 16891$. When $T_a = 1.03$, $T_b = 0.63$ and $T_p = 0.59$, the optimal solution $G(\boldsymbol{T}) = 1.43 \times 10^7$ is achieved.

Let $C_0 = 18891$. When $T_a = 0.47$, $T_b = 1.4$ and $T_p = 0.48$, the optimal solution $G(\boldsymbol{T}) = 0.68 \times 10^7$ is achieved.

In Option 1 in the above examples, ρ is selected to balance the effect of the expected cost and the variance of the cost. If the variance is small, compared with the expected cost, a large ρ is selected to highlight the influence of variance. If a small ρ is selected, the variance would have little influence on the optimal decision. On the other hand, if the variance is large, a small ρ is selected to emphasize the effect of expected cost. Because with the δ method, the number of systems is 200; while with the bootstrap method, the number of systems is 10. There exists a quadratic relationship between the total variance and the number of systems. The total variance with δ method is much larger than that with bootstrap method, a smaller ρ is selected. In reality, ρ may be selected on the basis of the attitude of the decision maker.

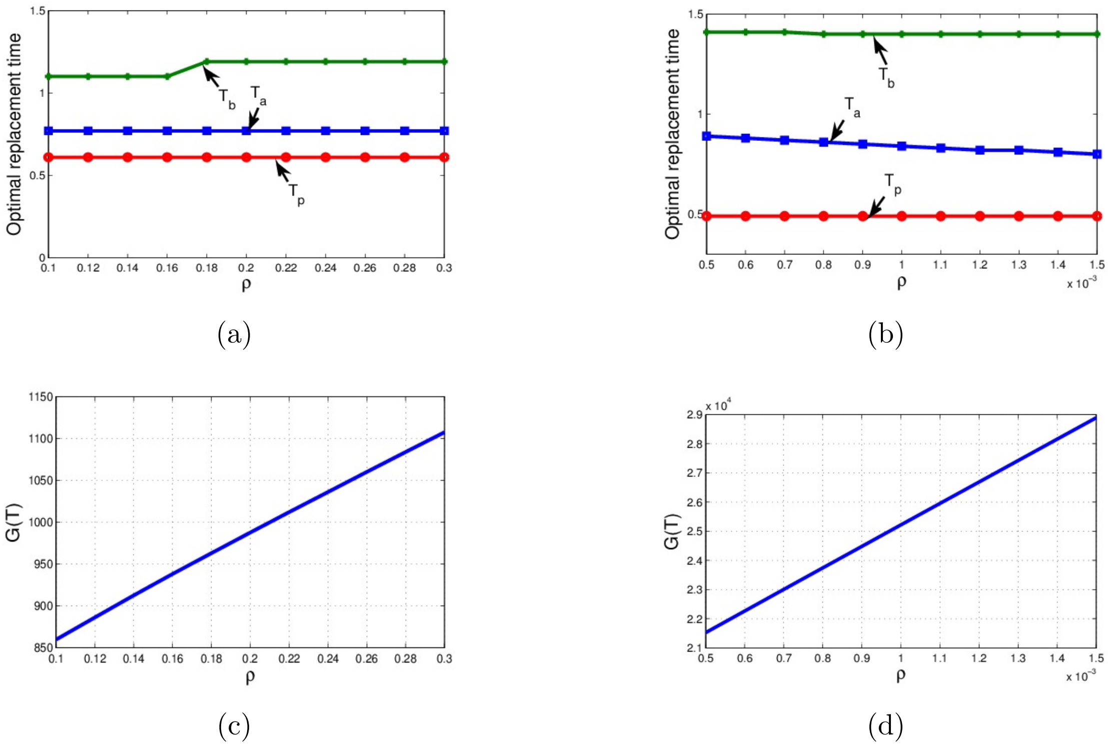

Fig. 1. Sensitivity analysis of ρ on the optimal decisions

The sensitivity analysis of ρ on the optimal decisions is shown in Figure 1. Optimal replacement time and maintenance cost is plotted with different ρ, with respect to the bootstrap method and δ method, respectively.

As shown in Figure 1, if ρ is increased towards 1, the objective function $G(T)$ presents an increasing trend (See (c) and (d) in Figure 1) and the optimal maintenance time does not share a similar behaviour (See (a) and (b) in Figure 1). This is because: the optimal maintenance time is determined jointly by the expected cost rate function and the variance function. As such, the relationship between the variance and maintenance time is not simply monotonously decreasing or increasing.

6 Conclusions

This paper presents the first attempt to collectively optimise maintenance policies for a set of different systems using the conditional value-at-risk theory, where model uncertainty is explicitly considered in optimising maintenance policy. The proposed approaches can especially be beneficial to the asset management sector (or warranty suppliers), in which a large number of different types of asset may be maintained and their reliability functions have different levels of efficiency. Collectively optimising preventive maintenance policies for each asset can be more cost-effective.

This paper provides practitioners with alternative methods to optimise maintenance policies: one can jointly optimise both the uncertainty of the expected cost of the maintenance policies considered for a portfolio of assets and the expected cost. The selection among the three options depends on decision maker's risk attitude.

We briefly discuss some open questions and possible implications for asset management.

Risk aggregation. The above discussion uses the variance as a measure of risk. Many other measures can also be applied. For example, the conditional value at risk (Rockafellar & Uryasev, 2000), the mean absolute deviation approach of Konno & Yamazaki (1991), the regret optimisation approach of Dembo & King (1992), and the minimax approach of Young (1998) are notable, and can be applied in the context of maintenance policy optimisation.

Adaptive preventive maintenance policy. As the above-mentioned, the reliability indexes are assumed to be estimated based on failure data before the first PM time point. With time development, more failure data can be collected and then fed into the model development. As such, the efficiency of the estimators and the PM policy can be improved.

Maintenance policies. In this paper, only periodic PM policy is discussed. With more data available over time, developing sequential PM policy is possible. Furthermore, one can also integrate predictive maintenance policies in the optimisation.

Other uncertainty. In this paper, we considered model uncertainty. Other sources of uncertainty may of course be taken into consideration. In developing maintenance policies for either one-unit systems or complex systems, cost information is needed. In reality, however, precise data on costs may not be available, which is especially true for complex systems which includes many units and cost of repairing the units vary.

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Appendix

Proof of Property 1.

Set the derivatives of $G(\boldsymbol{T})$ in respect to T_a, T_b, and T_p to zero, respectively, and obtain

$$\frac{\partial C_a(T_a|\boldsymbol{\theta}_a)}{\partial T_a} + \rho m_a \frac{\partial \mathrm{V}_{\boldsymbol{\theta}}(C_a(T_a|\boldsymbol{\theta}_a))}{\partial T_a} = 0 \tag{34}$$

$$\frac{\partial C_b(T_b|\boldsymbol{\theta}_b)}{\partial T_b} + \rho m_b \frac{\partial \mathrm{V}_{\boldsymbol{\theta}}(C_a(T_b|\boldsymbol{\theta}_b))}{\partial T_b} = 0 \tag{35}$$

$$\frac{\partial C_p(T_p|\boldsymbol{\theta}_p)}{\partial T_p} + \rho m_p \frac{\partial \mathrm{V}_{\boldsymbol{\theta}}(C_p(T_p|\boldsymbol{\theta}_p))}{\partial T_p} = 0 \tag{36}$$

Plugging Eq. (1) into Eq. (34), one has

$$\begin{aligned}(c_{a1} - c_{a2}) r_a(T_a|\boldsymbol{\theta}_a) \int_0^{T_a} \bar{F}_a(t|\boldsymbol{\theta}_a)\mathrm{d}t - (c_{a1} - c_{a2}) F_a(T_a|\boldsymbol{\theta}_a) \\ = c_{a2} - \mu m_a \frac{\mathrm{dV}_{\boldsymbol{\theta}}(C_a(\boldsymbol{T}_a|\boldsymbol{\theta}_a))}{\mathrm{d}T_a} \frac{\left[\int_0^{T_a} \bar{F}_a(t|\boldsymbol{\theta}_a)dt\right]^2}{\bar{F}_a(T_a|\boldsymbol{\theta}_a)}\end{aligned} \tag{37}$$

where $r_a(T_a|\boldsymbol{\theta}_a)$ represents the failure rate $f_a(T_a|\boldsymbol{\theta}_a)/\bar{F}_a(T_a|\boldsymbol{\theta}_a)$.

If one further assumes that $r_a(T_a|\boldsymbol{\theta}_a)$ is continuous and increasing, the left side of Eq. (37) is continuous and increasing. Since both $\mathrm{V}_{\boldsymbol{\theta}}(C_a(\boldsymbol{T}_a|\boldsymbol{\theta}_a))$ and $\left[\int_0^{T_a} \bar{F}_a(t|\boldsymbol{\theta}_a)dt\right]^2/\bar{F}_a(T_a|\boldsymbol{\theta}_a)$ are increasing in T_a, the right side of Eq. (37) is decreasing. The minimum value of the left side is 0 (when $T_a = 0$) and the maximum value of the right side of Eq. (37) is c_{a2} (when $T_a = 0$), so there exists a solution T_a^* satisfying Eq. (37).

Similarly, substituting Eqs (2) and (3) into Eq. (35) and Eq. (36), respectively, one obtains

$$c_{b1} m(T_b|\boldsymbol{\theta}_b) T_b - c_{b1} M(T_b|\boldsymbol{\theta}_b) = c_{b2} - \rho m_b \frac{\mathrm{dV}_{\boldsymbol{\theta}}(C_b(\boldsymbol{T}_b|\boldsymbol{\theta}_b))}{\mathrm{d}T_b} T_b^2 \tag{38}$$

and

$$c_{p1} \lambda(T_p|\boldsymbol{\theta}_p) T_p - c_{p1} \Lambda(T_p|\boldsymbol{\theta}_p) = c_{p2} - \rho m_p \frac{\mathrm{dV}_{\boldsymbol{\theta}}(C_p(\boldsymbol{T}_p|\boldsymbol{\theta}_p))}{\mathrm{d}T_p} T_p^2 \tag{39}$$

where $m(.)$ is the renewal density function and $\lambda(.)$ is the failure intensity function.

Similarly, one can prove that there exist solutions for Eq. (38) and Eq. (39).

This proves Property 1. □

Proof of Property 2.

One can write the Lagrange function for the optimisation problem in Eqs (10) and (11) as:

$$
\begin{aligned}
L(\boldsymbol{T}) = & m_a C_a(T_a|\boldsymbol{\theta}_a) + m_b C_b(T_b|\boldsymbol{\theta}_b) + m_p C_p(T_p|\boldsymbol{\theta}_p) \\
& + \mu(m_a^2 \mathrm{V}_{\boldsymbol{\theta}}(C_a(T_a|\boldsymbol{\theta}_a)) + m_b^2 \mathrm{V}_{\boldsymbol{\theta}}(C_b(T_b|\boldsymbol{\theta}_b)) + m_p^2 \mathrm{V}_{\boldsymbol{\theta}}(C_p(T_p|\boldsymbol{\theta}_p)) - \nu_0).
\end{aligned} \tag{40}
$$

Then, the first order necessary conditions for a solution in nonlinear programming to be optimum, or the Karush-Kuhn-Tucker conditions (see Zörnig (2014), p. 72), are given by those equations shown in Eqs (12)—(15), and the following two inequalities:

$$
m_a^2 \mathrm{V}_{\boldsymbol{\theta}}(C_a(T_a|\boldsymbol{\theta}_a)) + m_b^2 \mathrm{V}_{\boldsymbol{\theta}}(C_b(T_b|\boldsymbol{\theta}_b)) + m_p^2 \mathrm{V}_{\boldsymbol{\theta}}(C_p(T_p|\boldsymbol{\theta}_p)) \leq \nu_0 \tag{41}
$$

$$
\mu \geq 0 \tag{42}
$$

Below we discuss the existence of possible solutions of the above conditions.

Case 1. $\mu = 0$.

Conditions (12),(13), and (14) become

$$
\frac{\partial C_a(T_a|\boldsymbol{\theta}_a)}{\partial T_a} = 0 \tag{43}
$$

$$
\frac{\partial C_b(T_b|\boldsymbol{\theta}_b)}{\partial T_b} = 0 \tag{44}
$$

$$
\frac{\partial C_p(T_p|\boldsymbol{\theta}_p)}{\partial T_p} = 0 \tag{45}
$$

Solving equations (43), (44), and (45) is equivalent to finding the PM policy for each system respectively, and then checking whether the constraint in (41) holds or not.

Case 2. $\mu > 0$.

In this case, the optimisation problem becomes solving Eqs (12), (13), (14), and the following equation:

$$
m_a^2 \mathrm{V}_{\boldsymbol{\theta}}(C_a(T_a|\boldsymbol{\theta}_a)) + m_b^2 \mathrm{V}_{\boldsymbol{\theta}}(C_b(T_b|\boldsymbol{\theta}_b)) + m_p^2 \mathrm{V}_{\boldsymbol{\theta}}(C_p(T_p|\boldsymbol{\theta}_p)) = \nu_0 \tag{46}
$$

Similar to the proof process in Property 1, from Eqs (43), (44) and (45), one can obtain solutions T_a^*, T_b^* and T_p^*, which are functions of μ. Substituting T_a^*, T_b^* and T_p^* into Eq. (46), one can obtain an equation with respect to μ. If the equation can be solved, then optimum T_a^*, T_b^* and T_p^* can be obtained.

This proves Property 2. □

Proof of Property 3.

If C_0 is large enough, the constraint shown in Eq. (17) can be ignored. As $m_a^2 \mathrm{V}_{\boldsymbol{\theta}}(C_a(T_a|\boldsymbol{\theta}_a)) + m_b^2 \mathrm{V}_{\boldsymbol{\theta}}(C_b(T_b|\boldsymbol{\theta}_b)) + m_p^2 \mathrm{V}_{\boldsymbol{\theta}}(C_p(T_p|\boldsymbol{\theta}_p))$ is decreasing in T_a, T_b and T_p, its minimum can be achieved if T_a, T_b and T_p are infinity. This implies that no replacement is needed when C_0 is large enough.

The rest of the proof is similar to that of the proof of Property 2. $\square$